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Unstable BP-operations and typical formal groups

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Abstract

We prove that the functor taking a $\mathbb{Z}_{(p)}$-algebra R to the set of homomorphisms of p-typical formal group laws over R is represented by the ring of additive unstable BP-cooperations. This is analogous to a result of Landweber concerning stable BP-cooperations and strict isomorphisms of formal groups. Using this point of view we are able to reproduce the formulae of Boardman–Johnson–Wilson describing the additional structure in $QBP_*\boldsymbol{BP}_{2*}$. Finally we discuss the set of multiplicative additive unstable BP-operations and give a characterisation in terms of homomorphisms of formal group laws.

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1. Introduction

Unstable cohomology operations have received considerable attention in recent times and now have a firm grounding in the work of Boardman et al. [3] where a comprehensive study of the structure of unstable algebras is presented. This and the Ravenel–Wilson calculation of the Hopf ring for complex cobordism [5] provide the background for the current work. We begin by summarising what we need. Let $\boldsymbol{BP}_{2k}$ be the $2k$th space in the Ω-spectrum associated to the Brown–Peterson spectrum BP. We have BP^*-modules.

BP^*BP = stable BP-cohomology operations,
$BP^*\boldsymbol{BP}_{2*}$ = even unstable BP-cohomology operations,
$PBP^*\boldsymbol{BP}_{2*}$ = even additive unstable BP-cohomology operations.

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Here $PBP^*\boldsymbol{BP}_{2*}$ consists of the primitive elements of $BP^*\boldsymbol{BP}_{2*}$. We can also consider BP-homology and there are strong duality isomorphisms

$$BP^*BP = Mod_{BP_*}(BP_*BP, BP_*),$$

$$BP^*\boldsymbol{BP}_{2*} = Mod_{BP_*}(BP_*\boldsymbol{BP}_{2*}, BP_*),$$

$$PBP^*\boldsymbol{BP}_{2*} = Mod_{BP_*}(QBP_*\boldsymbol{BP}_{2*}, BP_*),$$

where $QBP_*\boldsymbol{BP}_{2*}$ is the module of indecomposables in $BP_*\boldsymbol{BP}_{2*}$. In what follows we regard $QBP_*\boldsymbol{BP}_{2*}$ as a $\mathbb{Z}_{(p)}$-algebra, the product coming from the Hopf ring circle product in $BP_*\boldsymbol{BP}_{2*}$.

Let R be a $\mathbb{Z}_{(p)}$-algebra and let

$TF(R)$ = the set of p-typical formal group laws over R,

$TI(R)$ = the set of triples (F, f, G) where $F, G \in TF(R)$ and f is a strict isomorphism from F to G,

$TH(R)$ = the set of triples (F, f, G) where $F, G \in TF(R)$ and f is a homomorphism from F to G.

The following statements are about ungraded $\mathbb{Z}_{(p)}$-algebras. Let $\mathscr{A}$ be the category of $\mathbb{Z}_{(p)}$-algebras and let $\mathscr{S}$ be the category of sets.

Theorem 1.1. (Araki [1]). *The functor* $TF: \mathscr{A} \to \mathscr{S}$ *is represented by the* $\mathbb{Z}_{(p)}$-*algebra* BP_*.

Theorem 1.2. (Landweber [4]). *The functor* $TI: \mathscr{A} \to \mathscr{S}$ *is represented by the* $\mathbb{Z}_{(p)}$-*algebra* BP_*BP.

Our main theorem is:

Theorem 1.3. *The functor* $TH: \mathscr{A} \to \mathscr{S}$ *is represented by the* $\mathbb{Z}_{(p)}$-*algebra* $QBP_*\boldsymbol{BP}_{2*}$.

We proceed to show that there are certain natural transformations giving rise to the usual additional structure maps of $QBP_*\boldsymbol{BP}_{2*}$. Using this point of view we can reproduce the explicit formulae of [3] describing these maps. This allows us to rephrase Theorem 1.3 in terms of a functor taking values in a richer category. We note that the natural transformation

$$TI \to TH$$

including strict isomorphisms into homomorphisms, corresponds to the inclusion

$$Alg_{\mathbb{Z}_{(p)}}(BP_*BP, R) \subset Alg_{\mathbb{Z}_{(p)}}(QBP_*\boldsymbol{BP}_{2*}, R)$$

induced by the stabilisation map

$$s: QBP_*\boldsymbol{BP}_{2*} \to BP_*BP.$$

We end with a discussion of multiplicative additive unstable BP-operations. A more general framework for these results is planned in joint work with Neil Strickland.

2. The algebra QBP_*BP_{2*} and the proof of Theorem 1.3

The algebra QBP_*BP_{2*} was understood by Ravenel and Wilson [5] (see also [3]). Let $A_{*,2*}$ be the free $BP_*[BP^*]$-Hopf ring on elements $\{b_i\}$ for $i \geq 0$ with $|b_i| = 2i$ and $b_0 = 1$ and with coproduct $\psi(b_i) = \sum_{m+n=i} b_m \otimes b_n$. Ravenel and Wilson show that the Hopf ring BP_*BP_{2*} is a certain quotient of $A_{*,2*}$. On passing to the indecomposable quotient we find that $A_{*,2*}$ has the structure of a commutative BP^*-algebra and a BP^*-bimodule. The canonical map (see [5]) $A_{*,2*} \to BP_*BP_{2*}$ induces a surjective BP^*-algebra homomorphism between indecomposable quotients

$$\rho : QA_{*,2*} \to QBP_*BP_{2*}.$$

Recall $BP_* = BP^{-*} = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ with $|v_i| = 2(p^i - 1)$. For $v \in BP^i$ write $\overline{[v]}$ for $[v] - [0_i] \in QBP_*BP_i$. We wish to consider $A_{*,2*}$ and BP_*BP_{2*} as $\mathbb{Z}_{(p)}$-algebras and we define $B \subset A_{*,2*}$ to be the sub-algebra generated by the elements b_i, so $B = \mathbb{Z}_{(p)}[b_1, b_2, \dots]$. By observing that $QBP_*[BP^*] \cong BP_* \otimes_{\mathbb{Z}_{(p)}} BP^*$ we can write

$$QA_{*,2*} = BP_* \otimes B \otimes BP^*.$$

Here the BP_* on the left corresponds to the elements v_i and the BP^* on the right to the elements $\overline{[v_i]}$. All tensor products are taken over $\mathbb{Z}_{(p)}$. Consider now the formal power series ring $BP_* \otimes B \otimes BP^*[\![s, t]\!]$. As usual we write $b(s) = \sum b_i s^i$. By slight, but obvious, abuse of notation we can consider the following elements in the above formal power series ring.

$$b\left(\sum a_{i,j} s^i t^j\right) \otimes 1 \tag{2.1}$$

and

$$1 \otimes \sum a_{i,j} b(s)^i b(t)^j, \tag{2.2}$$

where we are writing the universal p-typical formal group law F_{BP} as $F_{BP}(x, y) = \sum a_{i,j} x^i y^j$. The above elements could then also be (suggestively) written as

$$b(F_{BP}(s, t)) \otimes 1$$

and

$$1 \otimes F_{BP}(b(s), b(t)).$$

The coefficient of $s^i t^j$ in the difference

$$\left(b\left(\sum a_{i,j} s^i t^j\right) \otimes 1\right) - \left(1 \otimes \sum a_{i,j} b(s)^i b(t)^j\right)$$

gives an element of $BP_* \otimes B \otimes BP^*$. Let $I \subset BP_* \otimes B \otimes BP^*$ be the ideal generated by these elements. The following is a reformulation of a result in [3].

Lemma 2.1.

$$QBP_*\boldsymbol{BP}_{2*} \cong BP_* \otimes B \otimes BP^*/I.$$

Proof. The ideal I can be identified with the kernel of the canonical map

$$\rho: BP_* \otimes B \otimes BP^* = QA_{*,2*} \to QBP_*\boldsymbol{BP}_{2*}$$

by appealing to the main result in [5] and observing that (2.1) and (2.2) above are the reduction to indecomposable quotients of the elements used by Ravenel and Wilson to get their main relation. □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We construct a set theoretic map

$$\Phi: Alg_{\mathbb{Z}_{(p)}}(QBP_*\boldsymbol{BP}_{2*}, R) \to TH(R)$$

and show this to be bijection. First note that the map

$$\rho^*: Alg_{\mathbb{Z}_{(p)}}(QBP_*\boldsymbol{BP}_{2*}, R) \to Alg_{\mathbb{Z}_{(p)}}(QA_{*,2*}, R) = Alg_{\mathbb{Z}_{(p)}}(BP_* \otimes B \otimes BP^*, R)$$

injects since ρ is a surjection and $Alg_{\mathbb{Z}_{(p)}}(-, R)$ is a contravariant right exact functor. Now, homomorphisms of $\mathbb{Z}_{(p)}$-algebras

$$BP_* \to R$$

are in one-to-one correspondence with p-typical formal group laws over R and homomorphisms of $\mathbb{Z}_{(p)}$-algebras

$$B \to R$$

are in one-to-one correspondence with power series $\sum_{i \geq 1} a_i x^i \in R[\![x]\!]$. This last correspondence associates $\phi: B \to R$ with $\sum_{i \geq 1} \phi(b_i) x^i \in R[\![x]\!]$.

Let $\theta \in Alg_{\mathbb{Z}_{(p)}}(QBP_*\boldsymbol{BP}_{2*}, R)$ and define $F_\theta(x, y)$ to be the p-typical formal group law corresponding to

$$\rho^*\theta|_{BP_* \otimes 1 \otimes 1}: BP_* \to R,$$

define $G_\theta(x, y)$ to be the p-typical formal group law corresponding to

$$\rho^*\theta|_{1 \otimes 1 \otimes BP^*}: BP^* \to R,$$

and define $f_\theta(x)$ to be the power series corresponding to

$$\rho^*\theta|_{1 \otimes B \otimes 1}: B \to R.$$

We now define the map

$$\Phi: Alg_{\mathbb{Z}_{(p)}}(QBP_*\boldsymbol{BP}_{2*}, R) \to TH(R)$$

by

$$\theta \mapsto (F_\theta, f_\theta, G_\theta).$$

We first check that Φ is well defined. To see this consider the map of formal power series rings

$$BP_* \otimes B \otimes BP^*/I[\![s,t]\!] \to R[\![s,t]\!]$$

induced by $\theta \in Alg_{\mathbb{Z}_{(p)}}(\mathbb{Q}BP_*\boldsymbol{BP}_{2*}, R)$. It is clear from the above descriptions that under this map

$$b(F_{BP}(s,t)) \otimes 1 \mapsto f_\theta(F_\theta(s,t))$$

and

$$1 \otimes F_{BP}(b(s), b(t)) \mapsto G_\theta(f_\theta(s), f_\theta(t)).$$

Now the two elements on the left are equal so $f_\theta(F_\theta(s,t) = G_\theta(f_\theta(s), f_\theta(t))$ in $R[\![s,t]\!]$, showing f_θ is indeed a homomorphism between F_θ and G_θ.

Next we show Φ is injective. Let $\theta_1, \theta_2 \in Alg_{\mathbb{Z}_{(p)}}(QBP_*\boldsymbol{BP}_{2*}, R)$ and suppose $(F_{\theta_1}, f_{\theta_1}, G_{\theta_1}) = (F_{\theta_2}, f_{\theta_2}, G_{\theta_2})$. Then by definition $\rho^*(\theta_1) = \rho^*(\theta_2)$, but ρ^* is injective so $\theta_1 = \theta_2$.

Finally we show Φ is surjective. Let (F, f, G) be a triple in $TH(R)$ and let

$$\phi_F: BP_* \to R,$$

$$\phi_f: B \to R,$$

and

$$\phi_G: BP_* \to R$$

be the maps representing F, f, and G. Construct the map

$$\phi: BP_* \otimes B \otimes BP^* \xrightarrow{\phi_F \otimes \phi_f \otimes \phi_G} R \otimes R \otimes R \to R$$

and claim it factors through $QBP_*\boldsymbol{BP}_{2*}$. It suffices to show the ideal $I \subset BP_* \otimes B \otimes BP^*$ is in the kernel. We do this by showing that

$$(b(F_{BP}(s,t)) \otimes 1) - (1 \otimes F_{BP}(b(s), b(t)))$$

is in the kernel of the induced map of formal power series rings

$$\phi: BP_* \otimes B \otimes BP^*[\![s,t]\!] \xrightarrow{\phi_F \otimes \phi_f \otimes \phi_G} R \otimes R \otimes R[\![s,t]\!] \to R[\![s,t]\!].$$

Now

$$\phi((b(F_{BP}(s,t))\otimes 1)-(1\otimes F_{BP}(b(s),b(t))))=f(F(s,t))-G(f(s),f(t))=0$$

since f is a homomorphism of formal groups. This completes the proof of the theorem. □

Remark 2.2. All of the above can be done for MU by replacing '$\mathbb{Z}_{(p)}$-algebra' by 'commutative ring', 'p-typical formal group' by 'formal group' and 'BP' by 'MU'.

3. Structure maps

In [3] one can find a description of all the additional structure present in $QBP_*\boldsymbol{BP}_{2*}$. Explicit formulae are given for the left and right units η_L and η_R, the counit, ε and the coproduct ψ. We show how to reproduce these in the spirit of Landweber from the above point of view. We can define natural maps:

$$\eta_L: TH(R)\to TF(R) \quad (F,\phi,G)\mapsto F,$$

$$\eta_R: TH(R)\to TF(R) \quad (F,\phi,G)\mapsto G,$$

$$\varepsilon: TF(R)\to TH(R) \quad F\mapsto (F,\mathrm{id},F),$$

$$\psi: TH^2(R)\to TH(R) \quad ((F,\phi,G),(G,\theta,H))\mapsto (F,\theta\phi,H),$$

turning the pair (TF, TH) into a semi-groupoid. (This is just a fancy name for a small category, but we stick with it to keep in line with our other objects.) Here $TH^2(R)$ is defined to be the set $\{(F,f,G)\times(F',f',G')\in TH(R)\times TH(R)\mid G=F'\}$. The above maps give us homomorphisms:

$$\eta_L: BP_*\to QBP_*\boldsymbol{BP}_{2*},$$

$$\eta_R: BP_*\to QBP_*\boldsymbol{BP}_{2*},$$

$$\varepsilon: QBP_*\boldsymbol{BP}_{2*}\to BP_*,$$

$$\psi: QBP_*\boldsymbol{BP}_{2*}\to QBP_*\boldsymbol{BP}_{2*}\otimes QBP_*\boldsymbol{BP}_{2*},$$

turning $(BP_*, QBP_*\boldsymbol{BP}_{2*})$ into a Hopf semi-algebroid over $\mathbb{Z}_{(p)}$, i.e. a semi-groupoid in the category of $\mathbb{Z}_{(p)}$-coalgebras.

Proposition 3.1 (Boardman et al. [3]). *Explicit formulae for the structure maps in $QBP_*\boldsymbol{BP}_{2*}$ are given by*

$$\eta_L(v_i)=v_i, \tag{3.1}$$

$$\eta_R(v_i)=\overline{[v_i]}, \tag{3.2}$$

$$\varepsilon(1)=1, \tag{3.3}$$

$$\varepsilon(b_i) = 0 \quad \text{for } i > 0, \tag{3.4}$$

$$\varepsilon(v_i) = v_i, \tag{3.5}$$

$$\varepsilon(\overline{[v_i]}) = v_i, \tag{3.6}$$

$$\sum_{k \geq 1} \psi(b_k)x^k = \sum_{i \geq 1} \left(\sum_{j \geq 1} b_j x^j \right) \otimes b_i. \tag{3.7}$$

This last formula is to be thought of as a formal power series equality in the formal variable x. In the usual way the coefficients of x give the desired result.

Proof. To prove (3.1) observe that η_{L} must satisfy the following property. If

$$F_\theta \otimes f_\theta \otimes G_\theta : QBP_* \boldsymbol{BP}_{2*} \to R$$

represents $(F, f, G) \in TH(R)$, then

$$(F_\theta \otimes f_\theta \otimes G_\theta) \circ \eta_{\mathrm{L}} : BP_* QBP_* \boldsymbol{BP}_{2*} \to R$$

represents F. It is clear that the left inclusion satisfies this condition, proving (3.1).

The proof of (3.2) is similar.

To prove (3.3)–(3.6) observe that ε must satisfy the following. If $F_\theta : BP_* \to R$ represents $F \in TF(R)$ then $F_\theta \circ \varepsilon : QBP_* \boldsymbol{BP}_{2*} \to BP_* \to R$ represents (F, id, F). So we must have $F_\theta \circ \varepsilon(v_i) = F_\theta(v_i)$ and $F_\theta \circ \varepsilon(\overline{[v_i]}) = F_\theta(v_i)$ and $F_\theta \circ \varepsilon(1) = 1$ and $F_\theta \circ \varepsilon(b_i) = 0$ for $i > 0$. Eqs. (3.3)–(3.6) now follow.

To prove (3.7) observe that ψ must satisfy the following. If

$$(F_\theta \otimes f_\theta \otimes G_\theta) \otimes (G_\theta \otimes g_\theta \otimes H_\theta) : QBP_* \boldsymbol{BP}_{2*} \otimes QBP_* \boldsymbol{BP}_{2*} \to R$$

represents $((F, f, G), (G, g, H)) \in TH^2(R)$ then

$$((F_\theta \otimes f_\theta \otimes G_\theta) \otimes (G_\theta \otimes g_\theta \otimes H_\theta)) \circ \psi : QBP_* \boldsymbol{BP}_{2*} \to R$$

represents (F, gf, G). Take R to be $QBP_* \boldsymbol{BP}_{2*} \otimes QBP_* \boldsymbol{BP}_{2*}$ and take

$$F_\theta \otimes f_\theta \otimes G_\theta : QBP_* \boldsymbol{BP}_{2*} \to R = QBP_* \boldsymbol{BP}_{2*} \otimes QBP_* \boldsymbol{BP}_{2*}$$

to be the left inclusion (so $f(x) = \sum_{i \geq 0} (b_i \otimes 1)x^i$) and take

$$G_\theta \otimes g_\theta \otimes H_\theta : QBP_* \boldsymbol{BP}_{2*} \to R = QBP_* \boldsymbol{BP}_{2*} \otimes QBP_* \boldsymbol{BP}_{2*}$$

to be the right inclusion (so $g(x) = \sum_{j \geq 0} 1 \otimes b_j x^j$). In this case

$$(F_\theta \otimes f_\theta \otimes G_\theta) \otimes (G_\theta \otimes g_\theta \otimes H_\theta) = \mathrm{id}$$

and so ψ represents (F, gf, G). In particular from the definition of the power series gf, we have

$$gf(x) = \sum_{i \geq 0} (gf)_\theta(b_i)x^i \tag{3.8}$$

$$= \sum_{i \geq 0} \psi(b_i)x^i. \tag{3.9}$$

However,

$$gf(x) = g(f(x)) \tag{3.10}$$

$$= \sum_{j \geq 0} 1 \otimes b_j(f(x))^j \tag{3.11}$$

$$= \sum_{j \geq 0} \left(\sum_{i \geq 0} b_i x^i \right)^j \otimes b_j, \tag{3.12}$$

giving the result. □

In view of the above discussion we can combine Theorems 1.1 and 1.3 to give the following.

Theorem 3.2. *The functor taking the $\mathbb{Z}_{(p)}$-algebra R to the semi-groupoid $(TF(R), TH(R))$ is represented by the Hopf semi-algebroid $(BP_*, QBP_*\boldsymbol{BP}_{2*})$.*

It is clear from the above and from Landweber's results on BP_*BP that the natural transformation

$$TI \to TH$$

given by the inclusion

$$TI(R) \hookrightarrow TH(R)$$

of strict isomorphism into homomorphisms, is induced by the stabilisation map $QBP_*\boldsymbol{BP}_{2*} \to BP_*BP$. Using the extra structure on $QBP_*\boldsymbol{BP}_{2*}$ and BP_*BP we see that the inclusion

$$(TF(R), TI(R)) \hookrightarrow (TF(R), TH(R))$$

is a map of semi-groupoids and so the stabilisation map is a map of Hopf semi-algebroids. This gives an alternative proof of the result in [3] that the stabilisation map is a map of BP_*-bimodules.

4. Multiplicative additive operations

Araki [2] has shown there is a bijection between the set of multiplicative stable BP-operations, $Mult(BP)$, and certain typical curves over F_{BP}. The most convenient form of his result for the present context is as follows.

Theorem 4.1. (Araki [2]). *There is a one-to-one correspondence*

$$Mult(BP) \simeq \{(F, f, G) \in TI(BP_*) \mid F = F_{BP}\}.$$

We shall extened this result to the unstable case. Given an additive unstable operation $\theta: BP^k(-) \to BP^k(-)$ we shall write θ_l for the looped operation $\Omega^{k-l}\theta: BP^{k-l}(-) \to BP^{k-l}(-)$.

Definition 4.2. An additive unstable operation $\theta: BP^k(-) \to BP^k(-)$ is said to be multiplicative if $\theta(xy) = \theta_j(x)\theta_i(y)$ for $x \in BP^i(X)$ and $y \in BP^j(X)$ with $i + j = k$. Let $UMult(BP) \subset PBP^*\boldsymbol{BP}_{2*}$ be the set of all such operations.

Recall that $PBP^*\boldsymbol{BP}_{2*}$ comes equipped with a coproduct $\psi_\circ$ given by $\psi_\circ(\theta) = \sum_{i+j=k}\psi_\circ^{ij}(\theta)$ where $\psi_\circ^{ij}: BP^*(\boldsymbol{BP}_k) \to BP^*(\boldsymbol{BP}_i \times \boldsymbol{BP}_j)$ is induced from the map $\boldsymbol{BP}_i \times \boldsymbol{BP}_j \to \boldsymbol{BP}_k$ coming from the ring structure in BP.

Lemma 4.3.

$$UMult(BP) = \{\theta \in PBP^*\boldsymbol{BP}_{2*} \mid \psi_\circ(\theta) = \sum_{i+j=k} \theta_j \otimes \theta_i\}.$$

Proof. The coproduct condition is equivalent to demanding that $\psi_\circ^{i,j}(\theta) = \theta_j \otimes \theta_i$ for all $i + j = k$. In other words the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{BP}_i \times \boldsymbol{BP}_j & \xrightarrow{\circ} & \boldsymbol{BP}_k \\ \downarrow{\scriptstyle \theta_j \times \theta_i} & & \downarrow{\scriptstyle \theta} \\ \boldsymbol{BP}_i \times \boldsymbol{BP}_j & \xrightarrow{\circ} & \boldsymbol{BP}_k \end{array}$$

This is precisely the diagram needed to show $\theta(xy) = \theta_j(x)\theta_i(y)$. □

We can now characterise $UMult(BP)$ in terms of formal group law homomorphisms as follows.

Proposition 4.4. *There is a one-to-one correspondence*

$$UMult(BP) \simeq \{(F, f, G) \in TH(BP_*) \mid F = F_{BP}\}.$$

Proof. Consider the set $Alg_{BP_*}(QBP_*\boldsymbol{BP}_{2*}, BP_*)$ of (left) algebra homomorphisms from $QBP_*\boldsymbol{BP}_{2*}$ to BP_*. By definition this is the set of $\theta \in Mod_{BP_*}(QBP_*\boldsymbol{BP}_{2*}, BP_*)$

such that there is commutative diagram

$$\begin{array}{ccc} QBP_*\boldsymbol{BP}_{2*} \otimes QBP_*\boldsymbol{BP}_{2*} & \xrightarrow{\theta \otimes \theta} & BP_* \otimes BP_* \\ \downarrow \circ & & \downarrow m \\ BP_*\boldsymbol{BP}_{2*} & \xrightarrow{\theta} & BP_* \end{array}$$

Using the duality $Mod_{BP_*}(QBP_*\boldsymbol{BP}_{2*}, BP_*) = PBP^*\boldsymbol{BP}_{2*}$ we can interpret θ as an element of $PBP^*\boldsymbol{BP}_{2*}$. As such, the above diagram is equivalent to the condition $\psi_\circ(\theta) = \sum_{i+j=k}\theta_j \otimes \theta_i$; thus by Lemma 4.3 there is a one-to-one correspondence

$$UMult(BP) \simeq Alg_{BP_*}(QBP_*\boldsymbol{BP}_{2*}, BP_*).$$

It is clear, however, from Theorem 1.3 that $Alg_{BP_*}(QBP_*\boldsymbol{BP}_{2*}, BP_*)$ is in one-to-one correspondence with the given subset of $TH(BP_*)$. □

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